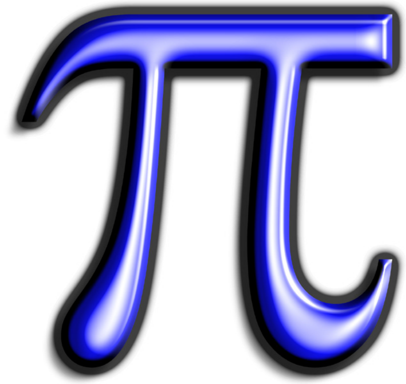
On Approximating *Pi*

by John M. Boyer, PhD, FSPE



1. Introduction

In mathematics, the Greek letter π, spelled *pi*, is associated with the numerical value of the ratio of the area of a circle to the square of the circle's radius as well as the ratio of a circle's circumference to its diameter.[1] Archimedes used a mathematical technique called the *method of exhaustion* to determine lower- and upper-bound values for *pi* that were the most accurate approximations of *pi* of any calculated before the Common Era.[2]

The method of exhaustion is typically used to estimate "the area of a shape by inscribing inside it [or surrounding it with] a sequence of polygons whose [combined] areas converge to the area of the containing [or surrounded] shape [as the number of polygons in the sequence increases]."[3] Recently, ISPE member Gary S. Flom presented a technique for approximating *pi* along with the question of whether anyone in ISPE had previously seen the method.[4] The purpose of this paper is to first explain the method of exhaustion and Archimedes' approach to approximating *pi*, then to answer Flom's question in the affirmative by situating his technique as an application of the method of exhaustion, and finally to briefly discuss limitations to and alternative methods for approximating *pi*.

2. Approximating *Pi* with the Method of Exhaustion

If a circle has a radius of 1 unit, then it is a *unit circle* with an area of π square units, because *pi* is the ratio of the area of a circle to the square of the circle's radius (which is 1). Therefore, approximating *pi* is the same as approximating the area of a unit circle.

As applied to approximating the area of a circle, and hence approximating *pi*, Figure 1 depicts the main idea of the method of exhaustion. In the left-hand diagram of Figure 1, the area within the black circle is approximated by the areas of the blue and purple hexagons. The area within the blue hexagon is a lower bound for the area within the black circle, because it has less area by six instances of the region labeled A. The upper bound for the area within the circle is given by the area of the circumscribing purple hexagon whose area is greater than the area within the black circle by six instances of the region labeled B. The area of the blue hexagon can be calculated by summing the areas of the triangles (3-sided polygons) labeled 1 through 6. Similarly, in the right-hand diagram of Figure 1, the area of the green octagon inscribed within the black circle can be calculated by summing the areas of the triangles labeled 1 through 8.

The key to understanding the method of exhaustion comes from comparing the left-hand diagram with the right-hand diagram. As the number of sides of the polygon inscribed within the black circle increases from 6 to 8, the number of regions within the area labeled A increases from 6 to 8, but the *size* of each individual region (each triangle) within the area labeled A is much smaller in the right-hand diagram than in the left-hand diagram. The net effect is that the octagon's area is a closer approximation of the circle's area—and of *pi*—than the hexagon's area. More generally, continued increases in the number of sides of the polygon inscribed within the black circle correspond with increasingly close approximations of the circle's area.

To verify that this is true, we must look further into the formula for calculating the area of polygons inscribed within a unit circle, such as the inscribed hexagon and octagon in Figure 1. According to the "Side-Angle-Side" trigonometric formula, the area of a triangle that has two sides of lengths a and b that form angle θ is $a \cdot b \cdot \sin(\theta) / 2$.[6] In both cases, the two triangle sides that join to form angle θ have a length of 1 because the black circle's radius is 1, so each triangle area is just $\sin(\theta) / 2$. In the left-hand diagram, the angle θ is 60° (which is 360° divided by six sides of a hexagon), and $\sin(60°) / 2$ is about 0.433.

There are six such triangles in the hexagon, so its total area is $0.433 \cdot 6 = 2.598$. This is not a great estimate for *pi*, because the areas of the regions labeled A are still quite large. In the right-hand diagram, θ is $360° / 8 = 45°$, so the area of the octagon is $8 \cdot \sin(45°) / 2 = 2.828$. This is still not a great estimate, but the estimation clearly moved in the proper direction as the number of sides of the inscribed polygon increased from 6 to 8.

More importantly, based on these calculations for the inscribed hexagon and octagon, it is easy to generalize to the following area formula for n-sided inscribed regular polygons:

[1] $n \cdot \sin(360°/n) / 2$

Now we can use Formula [1] above with larger values of n to see lower-bound approximations of *pi* that are closer to the actual value. For $n = 96$, Formula [1] yields a result of 3.1394, and evaluating Formula [1] for $n = 192$ yields 3.1410. These values show that the method of exhaustion works because as the value of n increases, the results approach the actual value of *pi*, which is just a little more than 3.14159. However, these two particular values of n were selected because they most clearly illustrate that Formula [1] is not exactly what Archimedes used.

3. Archimedes' Approximations of *Pi*

In the third century BCE, Archimedes produced lower- and upper-bound estimates on the value

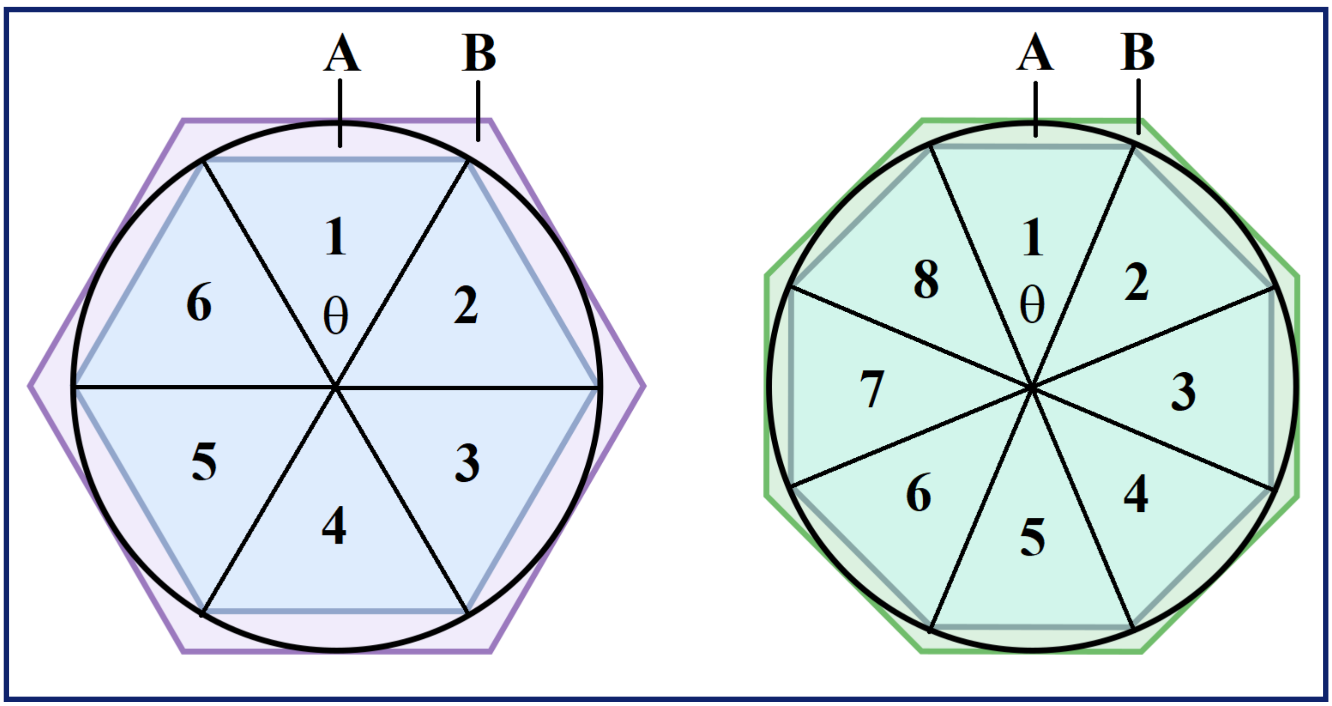


Figure 1. The method of exhaustion can be used to approximate lower- and upper-bound values for the area of a circle. As the numbers of sides of the polygons increase, the lower- and upper-bound approximations become closer to the actual area of the circle. This figure is an edited version of a public-domain file.[5]

of *pi* that were approximately 3.1408 and 3.1429, respectively.[7] His analyses were based on 96-sided regular polygons, or 96-gons, inscribed within and circumscribing a unit circle.[8] Note that although his lower-bound calculation is based on a 96-gon, his estimate is much closer to, but not equal to, the lower bound that we get from Formula [1] with $n = 192$, i.e., a 192-gon. There are two reasons for the discrepancies. The first reason has to do with the way Archimedes applied the method of exhaustion to produce these estimates. The second reason has to do with the way Archimedes had to calculate numerical values as he applied the method of exhaustion.

The method of exhaustion was developed two centuries before Archimedes and was used to calculate areas and volumes.[9] Archimedes applied the method of exhaustion to compute the area of a circle, but he did so indirectly, by using inscribed and circumscribing polygons to show that the area of a circle is πr^2, where π (*pi*) is *defined* to be the ratio of a circle's circumference *C* to its diameter *d* ($\pi = C/d$).[10] He then applied the method of exhaustion directly to measure the perimeters of the inscribed and circumscribing regular polygons, rather than their areas, to obtain lower- and upper-bound estimates of the circumference of a unit circle, half of which is, then, an estimate of *pi* ($\pi = C/d = C/2r = C/2$).

Figure 2 depicts how to process each triangle of a regular polygon inscribed in a unit circle to determine the triangle's contribution to the perimeter of the polygon. Each triangle *T* can be split into two right triangles by bisecting the angle θ. The hypotenuse of each right triangle has length 1, the radius of the unit circle. Since the sine function is defined to be "opposite over hypotenuse,"[11] each right triangle's contribution to the polygon's perimeter is $\sin(\theta/2)$, so the triangle *T* contributes $2 \cdot \sin(\theta/2)$ to the perimeter of the inscribed regular polygon. Recall that the value of θ in an *n*-gon is $360°/n$, so triangle *T* contributes $2 \cdot \sin(180°/n)$. We then multiply by *n* for the *n* triangles in an inscribed regular *n*-gon, and then divide by 2 to convert the calculation of the perimeter of the *n*-gon into the lower-bound estimate of *pi* in Formula [2]:

[2] $n \cdot \sin(180°/n)$

Using Formula [2] with a 96-gon, we get the same result as we got for Formula [1] with a 192-gon. More to the point, evaluating Formula [2] with $n=96$ gives the result of 3.1410, which clears up most of the discrepancy with Archimedes' lower-bound estimate for *pi* of 3.1408. The small remaining discrepancy is due to the fact that Archimedes could not directly evaluate the sine function; and so, instead, he had

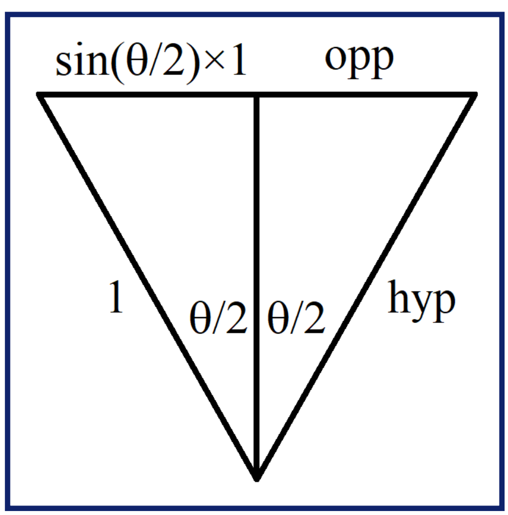


Figure 2. Depiction of the contribution each triangle makes to the perimeter of the inscribed regular n*-gon that contains it.*

to do a great deal of work iteratively using half-angle trigonometric identities and using slightly lower *estimates* of certain quantities.[12]

4. Flom's *Pi*-Approximation Technique

Whereas Archimedes used the method of exhaustion with triangle areas in a proof relating a circle's area to its circumference, and hence *pi*, Flom used the method of exhaustion with triangle areas to directly approximate *pi*.[13] Flom's generalized formula for *n*-gons is presented below as Formula [3]:

[3] $n \cdot \sin(180°/n) \cdot \cos(180°/n)$

Using n=192, Formula [3] produces the value 3.1410, which is the same result as Formula [1] with n=192. In fact, although Formula [3] is longer, it can be equated with Formula [1] by applying the double-angle formula $\sin(2\theta) = 2 \cdot \sin(\theta) \cdot \cos(\theta)$.[14] The key difference is that the triangle area was computed using the "Side-Angle-Side" trigonometric formula in Formula [1], whereas Flom's version is based on the well-known triangle area formula $A = ½ \cdot \text{base} \cdot \text{height}$,[15] combined with trigonometric definitions of sine and cosine.[16] Figure 3 helps to illustrate the triangle area calculation upon which Formula [3] is based.

In Figure 3, we can see that half of the triangle base is given by the same quantity that Archimedes used to calculate half of the triangle's contribution to the perimeter of its containing *n*-gon. The line that bisects the triangle into two right triangles is also the side adjacent to the half-angle $\theta/2$. Since cosine is defined to be adjacent over hypotenuse, and the hypotenuse is 1, the triangle height can be calculated using $\cos(\theta/2)$. Recalling that $\theta = 360°/n$, Formula [3] follows, i.e., there are n triangles, each with area (base / 2) • height = $\sin(180°/2) \cdot \cos(180°/n)$.

Comparing Formula [3] with Archimedes' Formula [2], one can see that Formula [3] contains Formula [2] and an extra factor, namely, $\cos(180°/n)$. This extra factor slows Formula [3]'s convergence toward *pi*. For example, Formula [2] requires a polygon with only n= 96 sides to calculate that *pi* must be at least 3.1410, but Formula [3] requires n=192 to establish the same lower bound. As n approaches ∞, the argument to cosine approaches 0, which means $\cos(180°/n)$ approaches 1, and Formula [3] approaches Archimedes' Formula [2]. However, for practical values of n, Formula [2] would have been preferable to Archimedes, due to having to evaluate fewer trigonometric functions and due to its faster convergence toward the value of *pi*

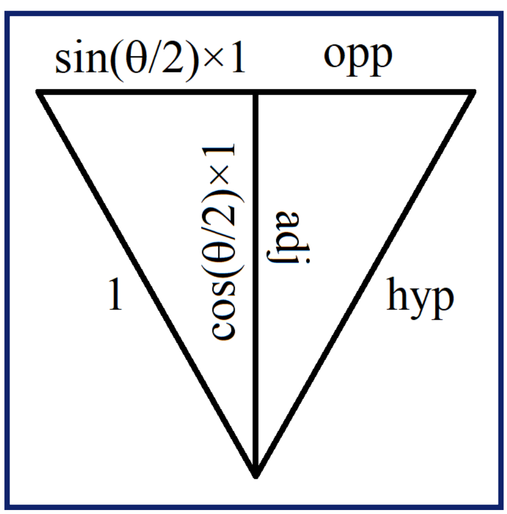


Figure 3. Rather than computing triangle area with the side-angle-side formula, it can be computed as one half of the base times the height of the triangle, where the base and height can be determined by evaluating the sine and cosine trigonometric functions.

(which translates to one less iteration of half-angle trigonometric identities to get the same estimate).

5. Beyond the Polygon

The parameter *n* in Formulae [1, 2, 3] allows us to increase the accuracy of our estimate of *pi* by increasing the number of sides *n* of the inscribed regular *n*-gon, thereby *exhausting* the size of the region-A gaps (see Figure 1) between the area of the *n*-gon's area and the containing circle's area. However, this ability to increase the estimate accuracy is not that useful, given the current devices we would use to evaluate Formulae [1, 2, 3]. Whether one uses computer-language commands, a spreadsheet, a calculator app, or a calculator, the current numerical accuracy typically available is 15 to 17 digits.[17] By experimenting with Archimedes' Formula [2] on a device of one's choosing, one can find that an *n* of around 90 million gives an estimate for *pi* that is accurate to 16 digits. Larger values of *n* won't improve accuracy on the selected device because of the numerical accuracy limit, and smaller values of *n* don't give a value for *pi* that is as accurate as the value one can get from the *pi* function on the selected device. Therefore, it is reasonable on typical devices to replace *n* with the value 90 million, as in Formula [4].

[4] $90{,}000{,}000 \cdot \sin(180°/90{,}000{,}000)$

There are computer-software libraries that enable computing of numerical values with much greater than the typical 15 to 17 digits of accuracy.[18] Although one could use larger values of *n* in Formula [2] with these libraries, one could instead use a newer method that can generate more digits of *pi* with less computation. For example, Formula [5] below, which uses radians rather than degrees, was developed in the early 1700s based on infinite series rather than on polygon areas or perimeters, and it was used to calculate *pi* to over 3,000 digits of accuracy in the 1950s.[19]

[5] $16 \cdot \arctan(1/5) - 4 \cdot \arctan(1/239)$

6. Conclusion

Archimedes originated the use of the method of exhaustion to estimate *pi*. His work included reasoning about the area of a circle based on the areas of polygons inscribed within the circle, and he directly estimated *pi* by taking half of the sum of the perimeter contributions of triangles within polygons inscribed within the unit circle. Flom's technique applies the method of exhaustion to estimate *pi* by summing the areas of triangles within polygons inscribed within the unit circle. Although using the method of exhaustion on polygons dominated the efforts to estimate *pi* for over 1,000 years after Archimedes, modern efforts use other approaches, such as infinite series formulae that converge more quickly to the value of *pi*.[20]

NOTES

1. Wikipedia contributors, "Pi," *Wikipedia, The Free Encyclopedia*, https://en.wikipedia.org/wiki/Pi.

2. Wikipedia contributors, "Approximations of π," *Wikipedia, The Free Encyclopedia*, https://en.wikipedia.org/wiki/Approximations_of_%CF%80.

3. Wikipedia contributors, "Method of Exhaustion," *Wikipedia, The Free Encyclopedia*, https://en.wikipedia.org/wiki/Method_of_exhaustion.

4. Gary S. Flom, "A Novel Way to Compute *Pi*," *Telicom* 33, no. 2 (April-June 2021): 69.

5. Wikipedia contributors, “File:Archimedes pi.svg,” *Wikipedia, The Free Encyclopedia*, https://en.wikipedia.org/wiki/File:Archimedes_pi.svg.

6. Wikipedia contributors, “Triangle,” *Wikipedia, The Free Encyclopedia*, https://en.wikipedia.org/wiki/Triangle#Computing_the_area_of_a_triangle.

7. Wikipedia contributors, “Archimedes,” *Wikipedia, The Free Encyclopedia*, https://en.wikipedia.org/wiki/Archimedes#Mathematics.

8. Ibid.

9. Wikipedia contributors, “Method of Exhaustion.”

10. William Dunham, *Journey through Genius: The Great Theorems of Mathematics* (New York, NY: Penguin Books, 1990).

11. Wikipedia contributors, “Trigonometric Functions,” *Wikipedia, The Free Encyclopedia*, https://en.wikipedia.org/wiki/Trigonometric_functions.

12. Petr Beckman, *A History of π (PI)* (New York, NY: St. Martin’s Press, 1971).

13. Flom, “A Novel Way to Compute Pi.”

14. Wikipedia contributors, “List of Trigonometric Identities,” *Wikipedia, The Free Encyclopedia*, https://en.wikipedia.org/wiki/List_of_trigonometric_identities.

15. Wikipedia contributors, “Triangle.”

16. Wikipedia contributors, “Trigonometric Functions.”

17. Wikipedia contributors, “Double-Precision Floating-Point Format,” *Wikipedia, The Free Encyclopedia*, https://en.wikipedia.org/wiki/Double-precision_floating-point_format.

18. Wikipedia contributors, “Arbitrary-Precision Arithmetic,” *Wikipedia, The Free Encyclopedia*, https://en.wikipedia.org/wiki/Arbitrary-precision_arithmetic.

19. Beckman, *A History of π (PI).*

20. Wikipedia contributors, “Pi.” **Ω**