Some Consequences of Newton's Shell Theorems

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1. Introduction

Why do mathematicians solve unrealistic, idealized problems? I suppose the cynical answer is, "Because they *can*." However, in the case of truly great mathematicians such as Isaac Newton, perhaps we should suggest a more noble reason. Could it be because they have romantic leanings that make them want to ignore all of the ugly imperfections of the real world and focus, instead, on a make-believe universe where everything is absolutely perfect? If the elements of our solar system are not perfect spheres with faultless internal symmetry, let the practical people deal with the actual. Of greater interest is the elegance of an idealized solar system that reveals laws and regularities that stand behind all of the clutter and ugliness found in nature.

For whatever reasons, Isaac Newton proved two remarkable theorems, commonly called *shell theorems.* These theorems relate to the gravitational attraction experienced by a particle and produced by a perfectly spherical body with the perfect internal radial symmetry resulting from its perfect shell-like layers of a certain thickness and of a material with a certain mass density.

These shell theorems are usually described as applying to two different situations. One case is where the particle is inside the sphere; the other is where the particle is outside the sphere. Nicely written proofs of these theorems are given elsewhere and won't be reproduced here.[1] This essay will condense the two cases into one with the observation that the case of a particle being outside of the sphere can be treated as if the particle is still inside of an outer layer, one that has a zero-mass density and an infinite thickness.

We can then summarize the consequences of Newton's two theorems with the following:

> A particle located within a symmetrically layered sphere experiences a gravitational acceleration that is different from what you would expect. This different acceleration will be the acceleration that would be experienced if the mass were redistributed as follows: All of the mass located as close to the center as the particle, or closer to the center than the particle, has been concentrated at a point at the center, and all of the mass located farther from the center than the particle does not exist.

This allows us to express the magnitude of the gravitation acceleration acting on the particle and directed toward the center as:

$a(d) = GM(d)/d^2$, where:

G is Newton's gravitational constant,

$M(d)$ is the total of mass located at a distance, p, from the center such that $p \leq d$, and

d is the distance of the particle from the center.

Perhaps the most intriguing result of Newton's theorems is that if the sphere has a hollow cavity at its center—that is, if there exists a positive D where $M(d)$ is zero for all d such that $0 \leq d \leq D$—then a particle anywhere within the cavity will be weightless.

The imaginary problem that will be discussed in this essay is as follows: We will suppose that the particle is fired from some sort of particle gun that is located at the center of the sphere. The particle is fired at a velocity v_0 through a pinhole that passes through the center and that extends through every layer of the sphere in both the direction of the muzzle and in the opposite direction.

We will imagine the particle traveling through the pinhole, thereby increasing d as it appears in the equation above from its initial value of zero to some positive value. In doing so, it might pass through some mass, thereby making M(d) increase from zero to some positive value. When this happens, a(d) will become non-zero and will decelerate the particle.

As the particle travels along the pinhole, it increases d and quite likely M(d) also. This makes a(d) vary in a complicated way, but—while it is still at a positive distance from the center—it will always be negative. This will reduce the particle's velocity. We would expect that, in time, the velocity will be reduced to zero and then turned negative as the particle falls back down through the pinhole, thereby making its velocity more negative. When it reaches the center again, it will have the velocity of $-v_0$, which is its original speed but in the opposite direction.

The particle will continue in that direction, moving through the other half of the pinhole in a mirror-image manner to its earlier trip. Thus it will, in time, reverse its direction and fall toward the center back through the pinhole, arriving at the center again with its original velocity v_0, thus completing the first of an infinite number of cycles through the sphere.

There *is* another possibility, one that does not repeat the trips through the sphere endlessly. If the original velocity, v_0, is large enough, the particle could reach the beginning of the outer layer—that is, exit from the sphere—with a velocity equal to or greater than the sphere's *escape velocity*. In this case, the particle would continue in its original direction forever and never return to the sphere. This escape velocity case will be discussed later in this essay.

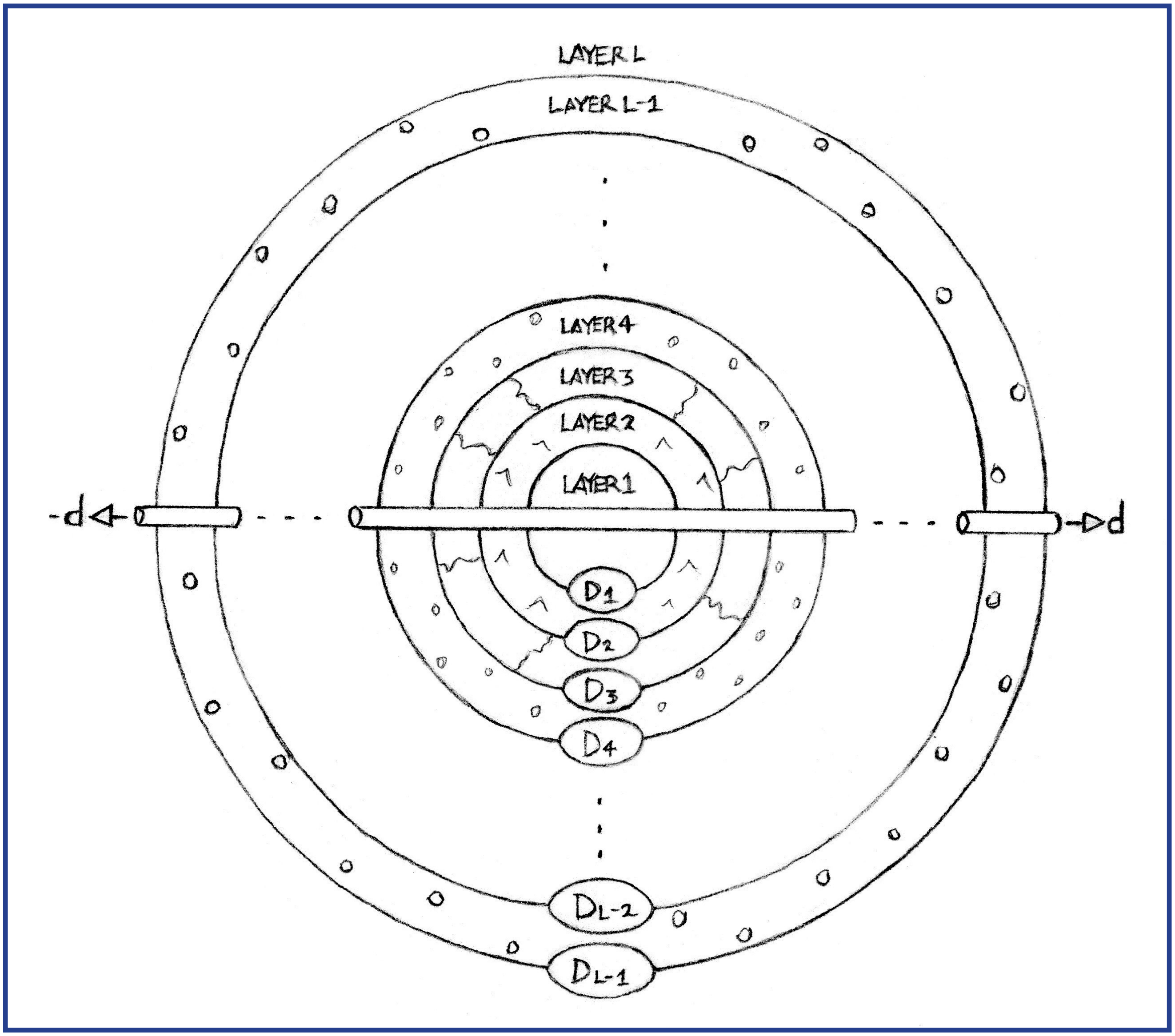
To provide a more detailed description of this idealized happening, we will exhibit a numerical procedure that produces a discrete approximation to the continuous mathematics we believe to be the mathematics used by nature in situations such as this. Our goal, then, will be to calculate quantities M_i, a_i, v_i, and d_i (at discrete points in time t_i) that are the values of *effective* mass, acceleration, velocity, and distance from the center of the sphere. We will start the procedure with i = 0 and work through i = 1, 2, 3, … until we either complete a cycle or escape the sphere. We will be given $d_0 = 0$, $M_0 = 0$, $D_0 = 0$, some value $v_0 > 0$, and the following description of the sphere:

> L = the number of layers; and for each layer, j = 1, 2, 3, …, L.
>
> D_j = the distance from the center of the sphere to the outer surface of layer j, where D_L is some distance that is larger than we expect the particle to reach if it doesn't escape, and μ_j = the mass density of the material in layer j, where μ_L is zero.

As usual, we will expect that our numerical approximation to the continuous problem will be improved upon by making the discrete increments in the independent variable small. This increases the solution effort, but if we can program a computer to do the calculations, we may not be very concerned with computational effort. So, let's say we have chosen the independent variable to be d, the distance of the particle from the center of the sphere. Let's also say that for numerical reasons, we would like the increment to be approximately Δ. It will be a convenience for the numerical procedure described below if we divide the thickness of each layer j into some number N_j of strata, each one a thickness, Δ_j, that is approximately Δ. We accomplish this by first computing an intermediate number $X_j = (D_j - D_{j-1})/\Delta$; and, if X_j is not integral, we round it up. Then we can compute a Δ_j for each layer as $\Delta_j = (D_j - D_{j-1})/X_j$, and an $N_j = N_{j-1} + X_j$, where $N_0 = 0$. We are given a number, INCNL, by which to increment N_L in case we underestimated D_L. (See the explanation of step A12 in Section 3 below.) We save the N_j and Δ_j for each j for use in the numerical procedure described in Section 3. But first, the sphere is illustrated.

2. A Cross-Section of the Layered Sphere

(Not shown are layers 5 through L-2. The pinhole width has been exaggerated.)



(*Illustration by Laurie McKnight*)

Figure 1: A Cross-Section of the Layered Sphere

3. The General Procedure A

With $D_0 = d_0 = t_0 = M_0 = a_0 = 0$ and v_0 = that given muzzle velocity, we start with i = 1 and j = 1 and execute the following steps of computations and logic:

A1: Set $d_i = d_{i-1} + \Delta_j$.

A2: Set $M_i = M_{i-1} + 4/3\pi\boldsymbol{\mu}_j(d_i^3 - d_{i-1}^3)$.

A3: Set $a_i = -GM_i/d_i^2$.

A4: Here we would like to set $v_i = v_{i-1} + 1/2(a_i + a_{i-1})\,\Delta_j/[1/2(v_i + v_{i-1})]$, but that equation has v_i on both of its sides. Therefore, we must first solve the equation $(v_i - v_{i-1})(v_i + v_{i-1}) = (a_i + a_{i-1})\Delta_j$ and then set $v_i = [v_{i-1}^2 + (a_i + a_{i-1})\Delta_j]^{1/2}$.

A5: Set $t_i = t_{i-1} + \Delta_j/[1/2(v_i + v_{i-1})]$.

A6: If $v_i \leq 0$, go to B1; otherwise, proceed.

A7: Increase i by one, and if $i \leq N_j$, return to step A1; otherwise, proceed.

A8: Increase j by one, and if j < L, return to step A1; otherwise, proceed.

A9: If j > L, go to step A11; otherwise, proceed.

A10: If $v_{i-1} \geq [(2GM_{i-1})/(D_{L-1})]^{1/2}$, stop; otherwise, return to step A1.

A11: Stop until told to proceed.

A12: Set j = L, $N_L = N_L$ + INCNL, and return to step A1.

4. Explanation of Procedure A

To explain procedure A, step A1 increases the distance of the particle from the center by the amount we have selected as the increment while in layer j. Step A2 increases the amount of mass that will have an effect on the particle by the mass density of layer j times the difference in volumes of a sphere of radius d_i and a sphere of radius d_{i-1}. (When j = L, there is no increase in the effective mass because μ_L is zero.) Step A3 calculates the acceleration of the particle produced by that increased mass.

Step A4 calculates the velocity of the particle attained by the time the particle reaches the *outer* surface of stratum i. To do this, it uses the average of the accelerations experienced at the beginning and ending of stratum i times the increment Δ_j, divided by the average of the velocities attained at the beginning and ending of stratum i. This results in the equation shown at step A4, which, unfortunately, has v_i on both of its sides. This requires the solution for v_i of the equation shown, and then, at last, v_i is set to that solution.

Step A5 updates the time at the end of stratum i using the increment for d while in layer j divided by the average of the velocities at the beginning and ending of stratum i.

Step A6 tests to see if the particle has stopped or turned around. If so, it exits the General Procedure A. Otherwise it proceeds to step A7 which increases the stratum index, i, by one and returns to A1 unless this increased stratum index, i, belongs to the next layer. In that case, step A8 increases the layer index j by one and then tests to see if the particle has just arrived at the beginning of layer L; that is, it has just reached the outer surface of the sphere. If not, the procedure returns to A1.

Step A9 tests to see if the particle failed to attain a non-positive velocity in layer L. If so, we jump to step A11. Otherwise, we continue with the procedure.

Step A10 tests to see if the velocity attained at the surface of the sphere (which is v_{i-1}, the velocity attained at the end of layer L-1) is equal to or greater than the sphere's escape velocity. If so, the procedure terminates with the particle continuing on forever. If not, the procedure returns to step A1 to process layer L, which is, if you recall, the vacuum of space outside of the sphere. The formula for the escape velocity results from equating the kinetic energy $1/2mv_{i-1}^2$

to the potential energy the particle would have if it were located at infinity.[2] That is, $1/2mv_{i-1}^2 = GM_{i-1}m/D_{L-1}$, where m is the mass of the particle L.

We do not expect to execute step A11, so the procedure stops to allow us to make a decision. The problem here is that we had set D_L to a distance that we believed was greater than the particle would reach while still having a positive velocity. But we are here because the particle's distance *did* exceed that D_L with the particle's velocity still positive. What is wrong? Did we underestimate D_L? Is there an error in the procedure? Did the solver make an arithmetical error? Should we quit? An alternative is to increase the number of strata in layer L and try again. Step A12 does just that.

5. The Turnaround Procedure B

This procedure will be executed if it is discovered in Procedure A that the particle will stop and reverse its direction. There is a bit of tidying up we might want to do before we continue with this procedure. This is because, in all likelihood, the velocity, v_i, will be less than zero, and the particle will have attained zero velocity at some place within stratum i. Perhaps we should compute a "special" $\Delta_s < \Delta_j$ such that if the computations A1, A2, A3, A4, and A5 were repeated with Δ_s, then A4 would produce a zero v_i and A5 would give us the time at which the zero velocity occurred. However, this appears to require an iterative procedure that does not seem to be worth the effort, especially if we have been using small values for Δ. (If you think otherwise and know a worthwhile way to accomplish the more exact result, please let me know.)

Instead, let's just interpolate linearly for an approximate Δ_s and repeat those calculations but fib a little about the new value of v_i. That is, let's execute:

B0: Set $\Delta_s = [v_{i-1}/(-v_i + v_{i-1})]\,\Delta_j$.

B1: Set $d_i = d_{i-1} + \Delta_s$.

B2: Set $M_i = M_{i-1} + 4/3\pi\mu_j(d_i^3 - d_{i-1}^3)$.

B3: Set $a_i = -GM_i/d_i^2$.

B4: Set $v_i = 0$.

B5: Set $t_i = t_{i-1} + \Delta_s/[1/2v_{i-1}]$.

So, now we have positioned the particle at its turnaround place and are prepared to do the calculations that follow the particle on its way back down to the center of the sphere. That is, we are in a position to develop the results for the last three quarters of the cycle that we now know will repeat itself indefinitely. It is not necessary to actually recalculate all of those results we computed for the first quarter of a cycle. We can just copy most of our previous calculations. Shown below are the instructions for doing that.

6. The Ending Procedure

We first save the index of the event when the velocity of the particle became zero. Let's call it IZERO, and then let's compute the total number of events there will be in a cycle. Let's call it ITOTAL. Thus:

Q1: Set IZERO = i and ITOTAL = 4IZERO, and then proceed.
  Here the second quarter feeds off the first quarter.
  Set I = IZERO + 1.
  Set k = IZERO – 1.

Q2: Set $d_i = d_k$.
  Set $M_i = M_k$.
  Set $a_i = a_k$.
  Set $v_i = -v_k$.
  Set $t_i = t_{i-1} + t_{k+1} - t_k$.
  Set i = i + 1.
  Set k = k - 1.
  If k > 0, return to Q2; otherwise, proceed.
  Here the third quarter also feeds off first quarter.
  Set k = 1.

Q3: Set $d_i = -d_k$.
  Set $M_i = M_k$.
  Set $a_i = -a_k$.

Set $v_i = -v_k$.
Set $t_i = t_{i-1} + t_k - t_{k-1}$.
Set $i = i + 1$.
Set $k = k + 1$.
If $k \leq$ IZERO, return to Q3; otherwise, proceed.
Here the fourth quarter feeds off the second quarter.
Set k = IZERO + 1. (This step is unnecessary, but it helps a reader.)

Q4: Set $d_i = -d_k$.
Set $M_i = M_k$.
Set $a_i = -a_k$.
Set $v_i = -v_k$.
Set $t_i = t_{i-1} + t_k - t_{k-1}$.
Set $i = i + 1$.
Set $k = k + 1$.
If $i \leq$ ITOTAL, return to Q4; otherwise, stop.

Here the cycle is complete. We will have all five arrays, d_i, M_i, a_i, v_i, and t_i, completely filled with data for i = 1, 2, 3, …, ITOTAL. Those data are intended to give a numeric approximation of the continuous functions that would be repeated endlessly in the idealized situation that has been imagined.

There are two other ways that this procedure could end. One is if the velocity of the particle when it first reaches the surface of the sphere is equal to or greater than the escape velocity. This stop occurs at step A10 as described in Section 3. If that stop occurs, those five data arrays will be filled up only to where the distance d_i equals D_{L-1}; that is, only up to the surface of the sphere.

The other ending is at step A11. If no errors have occurred, then the reason for this stop is that the procedure has reached the distance D_L without the velocity of the particle becoming non-positive. This is an awkward situation, since layer L really should have no finite D; but a hard lesson learned while programming mainframe computers was not to create procedures that permitted infinite loops. This procedure *does* allow the solver to increase D_L if he chooses to do so. Even though the problem dealt with in this essay is an extreme simplification of anything that might be found in the actual universe, I would be interested in learning of any improvement in, or correction to, the solution method that may have occurred to you. Better still would be an exact, continuous solution.

NOTES

1. Kansas State University Mathematics, "Newton's Shell Theorem," https://www.math.ksu.edu/~dbski/writings/shell.pdf.

2. Georgia State University HyperPhysics, "Escape Velocity," http://hyperphysics.phy-astr.gsu.edu/hbase/vesc.html. **Ω**