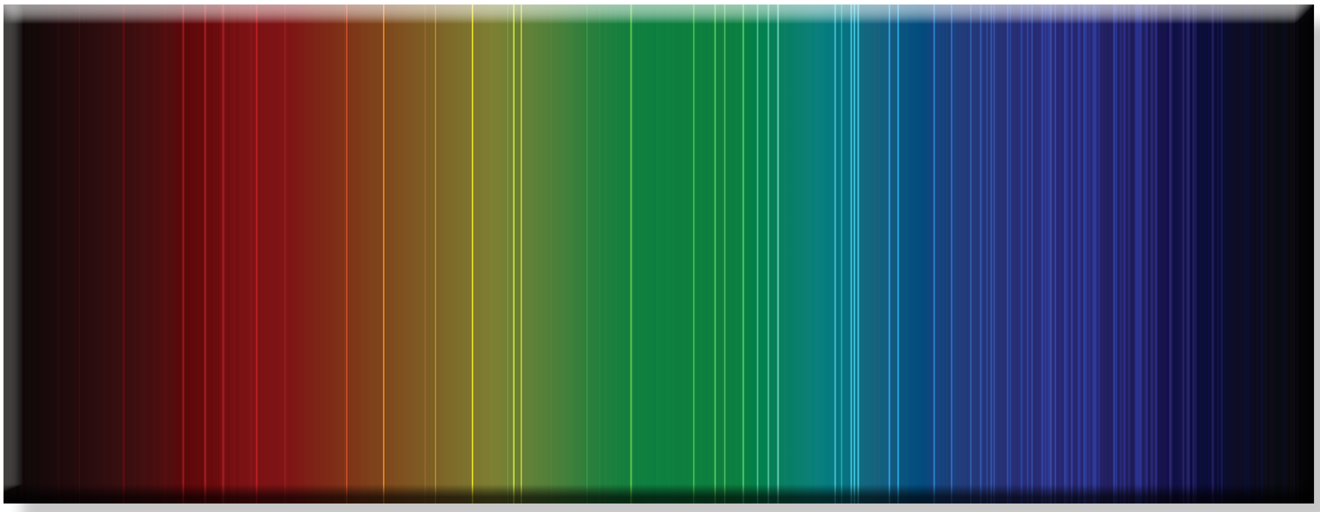
On the Science and Mathematics of Color Perception

by John M. Boyer, PhD, FSPE



Introduction

In an ISPE social media group post, a Fellow member of ISPE presented a "physics topic" question about why the primary and secondary colors form a color *wheel* (i.e., a circular shape) if the colors in the visible light spectrum are arranged *linearly* in the order red, orange, yellow, green, blue, indigo, and violet.[1] The question articulation included the following points:

1) The primary colors red and yellow combine to make the secondary color orange, and the wavelength of orange is between red and yellow in the spectrum.

2) The primary colors yellow and blue combine to make the secondary color green, and the wavelength of green is between yellow and blue in the spectrum.

3) The pattern of 1 and 2 above is broken with purple because the primary colors red and blue make purple, but purple (violet) has a shorter wavelength than blue and so it is *not* between red and blue in the visible light spectrum.

In the discussion thread of the post, an additional question arose about what structural changes would occur to the color wheel if we could see more than three primary colors. This article answers these questions by beginning with why violet violates *a* pattern, moving on to what *the* pattern is, then proceeding to the development of color "wheels" for higher numbers of primary colors.

Physics and Biology in Color Perception

Although purple is a special case in human vision, the first step in seeing why it is special is to dispel the belief in the ostensible pattern above that has made purple *seem* exceptional. From physics, we know that the visible light spectrum corresponds to linearly changing wavelengths. But a second fact from physics is that two waves of any kind that have different wavelengths do not combine to produce one wave with a wavelength between the two original wavelengths. In particular, visible light is comprised of discrete wave-particle duals called photons that don't directly combine with one another at all.[2] Therefore, purple doesn't defy a physics-based pattern of wavelength combination—there is no such physics-based pattern. Instead, perceptions of combined colors result from the human biological process described below.

Normal human eyes achieve sight using two kinds of sensors: rods and cones. Rods are used primarily for night vision, and cones are used for color vision in daylight. Human vision is called trichromatic because there are three types of cones: S-cones, M-cones, and L-cones. In the 1950s, George Wald performed experiments to determine the light sensitivities of our eye cones.[3] S-cones are sometimes called blue cones because they are most sensitive to shorter wavelengths of blue and violet light. M-cones are sometimes called green cones because they are most activated by green and yellow light in the middle of the wavelength range of visible light. The L-cones are sometimes called red cones and are most sensitive to longer wavelengths of visible light in the range from yellow to red light. Figure 1 is a visual summarization of Wald's work in which the solid blue, green, and red curves show the sensitivities of S-cones, M-cones, and L-cones relative to light wavelengths measured in nanometers (nm).[4]

As the diagram in Figure 1 illustrates, the light sensitivities of human eye cones are not matched well with the definition of primary colors being red, *yellow*, and blue. The reason is that red, *yellow*, and blue are the primary colors for mixing light-*absorbing* pigments, such as those used in painting, whereas the colors red, *green*, and blue are more commonly used as the primary colors of light-*emitting* systems, such as televisions and computer screens, because their light emissions are intended to be absorbed by the pigments in human eye cones.[5]

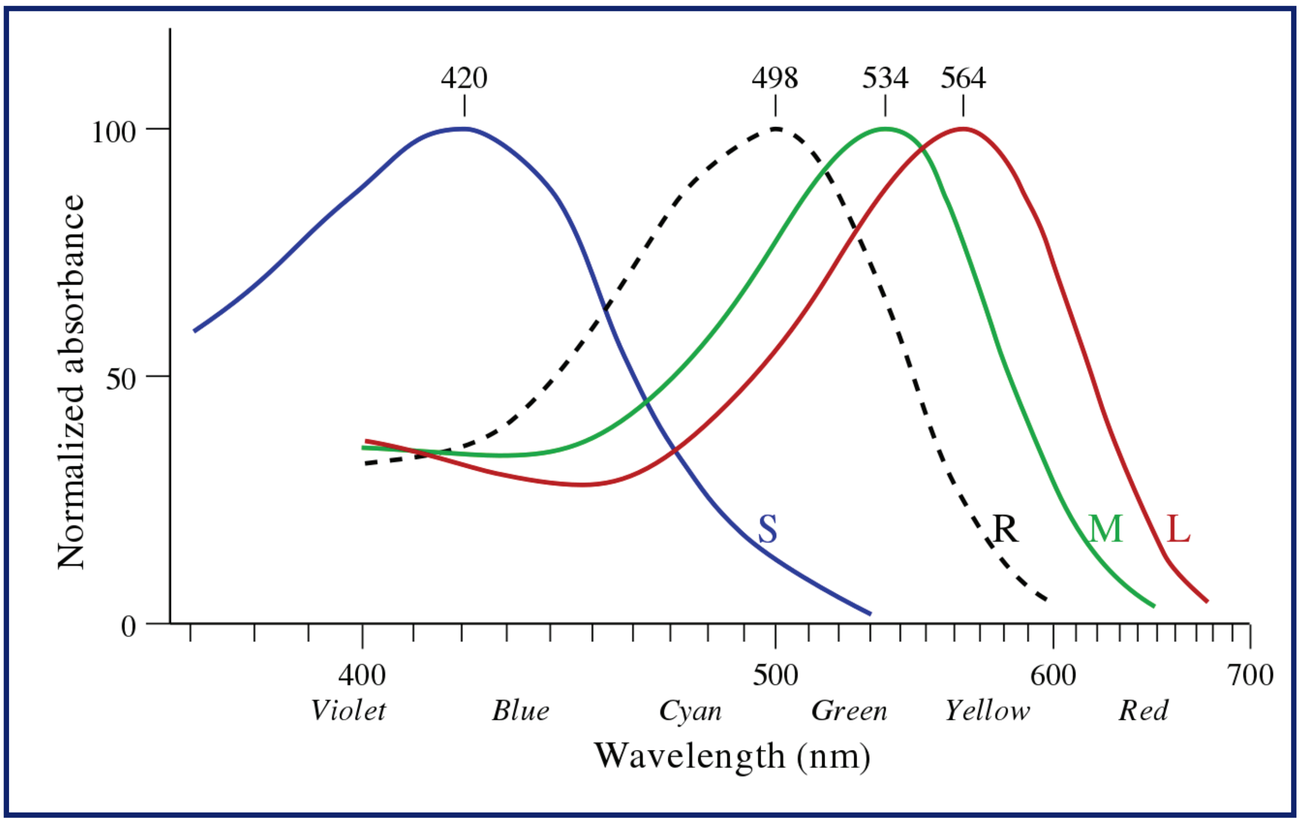


Figure 1. The light-wavelength absorption sensitivities of normal human eye cones and rods. The dotted curve is for rods, and the three solid curves are for the S-cones (max. at 420 nm), the M-cones (max. at 534 nm), and the L-cones (max. at 564 nm).[6]

Even with a change of primary colors, there is more to human color vision than just light absorption by pigments in three types of eye cones. The greatest sensitivities of the so-called "blue" and "green" cones are similar to but not quite the same as their color names: S-cones are most sensitive *between* blue and violet light, and the greatest sensitivity of M-cones is *between* green and yellow. However, the so-called "red" cones (L-cones) are most highly activated by *yellow* light, not red light.

In general, humans perceive most colors by detecting differences in the activation levels of different types of cones. For example, although L-cones are most highly activated by yellow light, they are still partly activated by red light, whereas M-cones are not. Hence, orange is perceived by detecting the amount of the difference between the activation levels of L-cones and M-cones. This neurological (biological) process applies not only to perceiving orange, but also to perceiving shades of red, yellow, green, cyan, blue, indigo, and violet.

Perceiving Purple

The neurological process of color perception loses the information about whether a differential in cone activation is due to one wavelength or more than one wavelength of light. For example, consider two methods for perceiving the color orange, based on Figure 1 and sample wavelengths of red, orange, and yellow.[7] If the orange color perception is the result of a single wavelength of orange light (e.g., about 600 nm), then the L-cones are much more highly activated than the M-cones. If the eyes are instead subjected to the combination of yellow light (e.g., about 570 nm) and red light (e.g., about 640 nm), then the yellow light activates the L-cones somewhat more than the M-cones, but the red light further augments the activation of the L-cones with little to no further activation of the M-cones. The net result is that the L-cones are more activated than the M-cones by approximately the same differential with red and yellow light as they were with orange light.

For many pairs of wavelengths, such as red and yellow, the neurological process of color perception produces the same result as if the eyes were receiving a single wavelength that is *between* the pair of wavelengths, such as orange. However, the combined perception of red and blue does not follow this pattern, since the wavelength range of violet (purple) is not between those of red and blue. Nonetheless, perceiving red and blue as purple is consistent with our neurological color-perception process when interpreted with Wald's results depicted in Figure 1.

From Figure 1, as monochromatic light wavelength decreases into the blue and then violet ranges, S-cone activation is high but decreases, M-cone activation is essentially constant, and L-cone activation is low but increases. The increase in activation of the pigment in L-cones in the violet light range is the key factor in why red and blue look like purple. For example, using Figure 1 and sample wavelengths of colors of the visible light spectrum,[8] a wavelength of blue (e.g., 450 nm) and a wavelength of red (e.g., 640 nm) produce activation levels for the S-cones and M-cones that are similar to the activation levels of S-cones and M-cones produced by a single wavelength of violet (e.g., 400 nm). For the violet wavelength, the activation of L-cones is higher than it is for the blue wavelength, but the red light further activates the L-cones only, with the net result that red and blue look like violet (purple).

Cone Combinatorics

Although the linear visible light spectrum maps into the shape of a color wheel due to increased activation of L-cones in the wavelength range of violet light, it is the mapping and not the shape that is surprising. A shape homeomorphic to a wheel, such as a triangle, would still be a natural representation for showing primary and

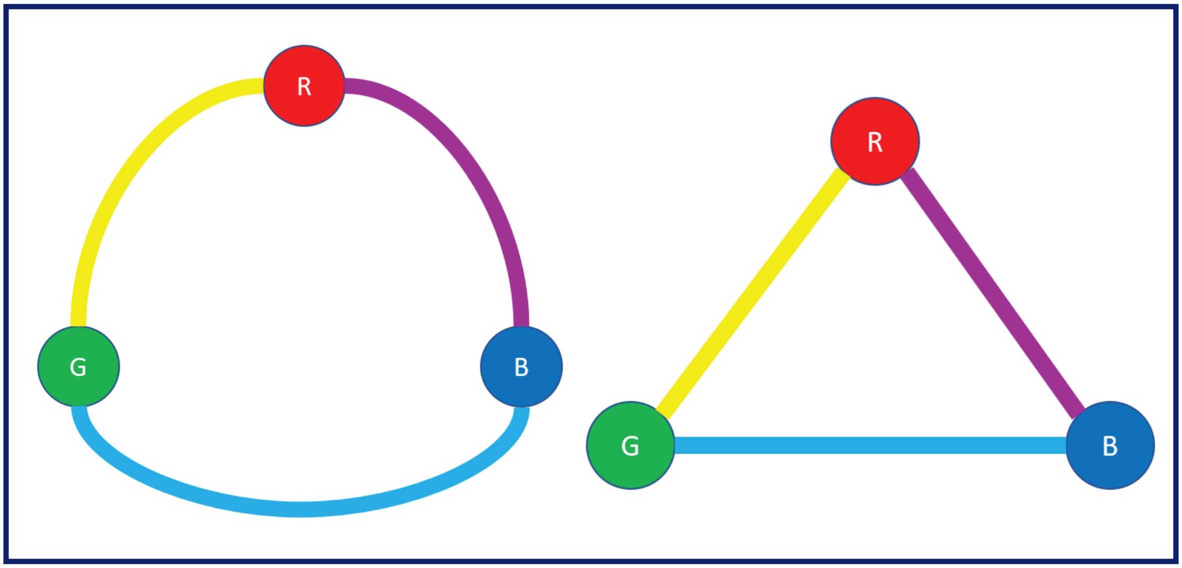


Figure 2. On the left is a trichromatic color wheel showing three vertices for three primary colors (red, green, and blue), three lines for secondary colors (yellow, cyan, and magenta), and internal and external regions for the presence or absence of all three primary colors. The color wheel is homeomorphic to a color triangle, which is shown on the right above.

secondary colors, even if human L-cones had no increased activation in the wavelength range of violet light. As shown in Figure 2, a wheel or triangle can have three vertices to represent the three primary colors; and it can have three edges (or curve segments) connecting pairs of vertices, each representing a secondary color that is the combination of the colors of its two vertex endpoints. For the edge connecting the vertices labeled R and B, humans may perceive its associated color as purple because of the extra L-cone activation region in the violet range; but if that activation region did not exist, then the R-B edge would still exist but would simply look different. Although it wouldn't be purple, it also wouldn't be just red nor just blue. Rather, it would look like whatever color name we would give to the simultaneous perception of red and blue.

In the same way, we use the color name "white" to refer to the simultaneous perception of all three primary colors. The color wheel also has a structural component for representing the perception of the color white: it has an internal region surrounded by the R-G-B cycle that can be associated with the simultaneously high activation of S-cones, M-cones, and L-cones. Similarly, the external region outside of the R-G-B cycle can be used to represent the absence of activation of any cones, which we perceive as the color black.

In the branch of mathematics called graph theory, a graph is a mathematical data structure comprised of a set V of vertices and a set E of edges, each of which has a pair of vertices from V as its endpoints. The graph in Figure 2 is called a *complete* graph on three vertices because there is an edge in E for every pair of vertices from V. The notation K_n is used as a shorthand for a complete graph on n vertices, so the complete graph in Figure 2 is denoted K_3.[9]

In the branch of mathematics called combinatorics, the binomial coefficients are positive integers that represent choosing k items from n items as calculated by the formula $n!/k!(n-k)!$, which is called the "n choose k" formula.[10] For the case of trichromacy, we have three primary colors, so n=3. Choosing zero of the three colors yields the color black, and there is only one way to make the choice: 3!/(0!(3-0)!)=1. The formula also tells us that there are three ways to choose k=1 color or k=2 colors from the n=3 colors, and the graph K_3 in Figure 2 shows the three primary colors as vertices and the three secondary colors as edges. The internal region surrounded by the cycle of vertices and edges in K_3 corresponds to the one way to choose three colors from a set of three primary colors, 3!/(3!(3-3)!) = 1; and the color white maps to choosing to activate all three primary colors. Note that the full two-dimensional space of Figure 2 is required to represent the four possible values of choosing k colors from n=3 colors.

These mathematical foundations are useful for answering the question of what a color "wheel" would look like for organisms whose color vision exceeds three distinct channels of color. For example, several species of birds, insects, and other animals have an additional type of eye cone for sensing ultraviolet light, so they have *tetrachromatic* vision.[11] When the number of primary colors increases to n=4, then the complete graph on four vertices, or K_4, can be used to help represent all the color combinations. In Figure 3, the vertices and edges of the K_4 are arranged into the shape of a *color sphere,* which is homeomorphic to the color tetrahedron shown on the right in Figure 3. The four vertices represent the colors red, green, blue, and ultraviolet. There are "4 choose 2" equals 6 edges because 4!/(2!(4-2)!) = (4×3×2×1)/((2×1)×(2×1)) = 6. Each edge represents one of the six secondary colors obtained by combining the pair of colors associated with the vertex endpoints of the edge, such as green-ultraviolet. The edges divide the sphere's surface into four regions, or *faces*, each homeomorphic to a triangle. Each triangular face can be used to represent the *tertiary* color that a tetrachromat perceives when its eyes simultaneously receive the three primary colors of the vertices along the border of the face. For example, the front-top face would represent the color red-green-ultraviolet, and the back-right face would represent the color red-blue-ultraviolet. The ball (volume of space) within the color sphere would correspond to white (choosing four of four colors), and the volume of space outside the color sphere would correspond to black (choosing zero of four colors).

There are a few observations about the trichromatic color triangle in Figure 2 and the

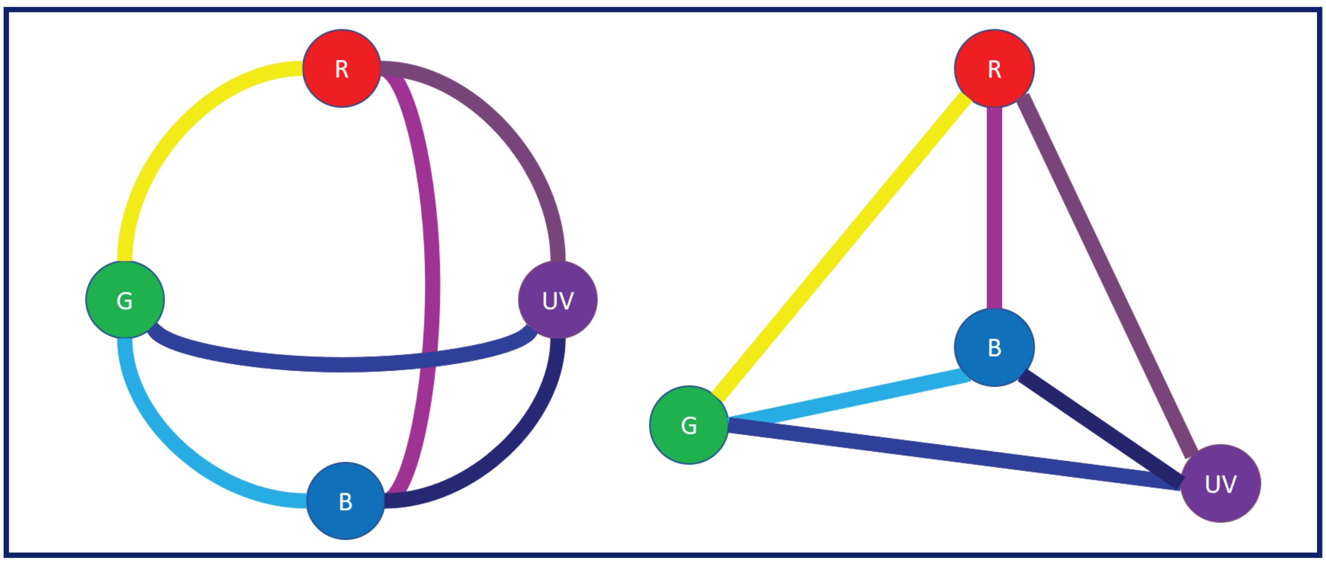


Figure 3. On the left is a color sphere representing color combinations for tetrachromats. The vertices and edges of a K_4 are arranged into a sphere that can represent all primary, secondary, and tertiary colors of tetrachromacy. The volume of space inside the sphere represents white, and the volume of space outside the sphere represents black. The color sphere is homeomorphic to a color tetrahedron, shown on the right. It is the higher-dimensional analog of the color triangle shown in Figure 2.

tetrachromatic color tetrahedron in Figure 3 that can help with the generalization of the multidimensional shape that can represent the basic color combinations of *n*-chromatic vision. For representing basic color combinations of trichromatic vision, a K_3 is embedded in two-dimensional space with its vertices and edges appearing along the perimeter of a triangle. For representing basic color combinations of tetrachromatic vision, a K_4 is embedded in three-dimensional space with its vertices and edges appearing along the surface of a tetrahedron. Using substitution of *n* to follow the pattern, for representing basic color combinations of *n*-chromatic vision, a K_n would be embedded in an "*n*-1"-dimensional space with its vertices and edges appearing along the surface of an "*n*-1"-dimensional simplex that is also sometimes called a hypertetrahedron.[12] For each value of *k* between 0 and *n*, the complete graph K_n has "*n* choose *k*" distinct complete subgraphs of the form K_k. For each value of *k*, each "*k*-1"-dimensional region in which each distinct K_k subgraph is embedded can be colored with the perception that results from combining the *k* primary colors associated with the *k* vertices along the boundary surface of the region.

The utility of hypertetrahedrons as a visual aid for showing color combinations decreases as dimensionality increases beyond our normal experience of three spatial dimensions. However, it is possible to see the generalization in action at one higher dimension, for pentachromatic vision.[13] As an example, consider a fifth type of eye cone—one that is activated by X-rays—that may be present in the citizens of the imaginary planet Krypton.[14] According to the generalization above, a K_5 would be embedded as a hypertetrahedron in four-dimensional space. It would be difficult even for Superman[15] to directly imagine a four-dimensional hypertetrahedron, but it is possible for both Superman and mere mortals to see most of it using the dimension-"flattening" technique that we have already used in Figure 3. The tetrahedron shown in Figure 3 is a three-dimensional object rendered on the page in a flattened, two-dimensional format. It looks like a triangle with an additional central vertex that forms additional triangles inside, and depth-perception cues trigger our ability to imagine the three-dimensional tetrahedron. Analogously, Figure 4 illustrates that a three-dimensional projection of a four-dimensional hypertetrahedron looks like a tetrahedron with an additional central vertex that forms additional tetrahedrons inside.

Using the diagram in Figure 4, we can iterate sequentially through the values of *k* from 0 to *n*=5 to see how the components of the tetrahedron can represent pentachromatic color combinations. The values *k*=0 and *k*=5 will be saved for last. Starting with *k*=1, there are five K_1 subgraphs that are embedded as vertices and that represent the five primary colors in Superman's eye cones. There are "5 choose 2" equals 10 K_2 subgraphs that are embedded as edges and that can each be colored with the secondary color perception representing the combination of the primary colors of the edge's endpoint vertices. There are "5 choose 3" = 5!/3!(5-3)! = 10 K_3 subgraphs, each of which is embedded as a triangle, whose internal region can be colored with the tertiary color perception obtained from combining the three primary colors of the vertices along the triangle's boundary.

The most interesting case that can be directly seen in Figure 4 occurs when *k*=4. There should be "5 choose 4" = 5!/(4!(5-4)!) = 5 K_4 subgraphs embedded as tetrahedrons and whose "internal" regions can each be colored with the quaternary color perception corresponding to the combination of the four primary colors of the vertices along the tetrahedron's boundary. It is easy to see how this can occur for four of the five tetrahedrons: the bottommost tetrahedron bound by XR, B, UV, and G; the front-facing tetrahedron bound by XR, R, G, and UV; the rightmost tetrahedron bound by XR, B, UV, and R; and a tetrahedron at the back of the diagram bound by XR, R, G, and B. There is a fifth tetrahedron bound by R, G, B, and UV, but its internal region appears to contain the other

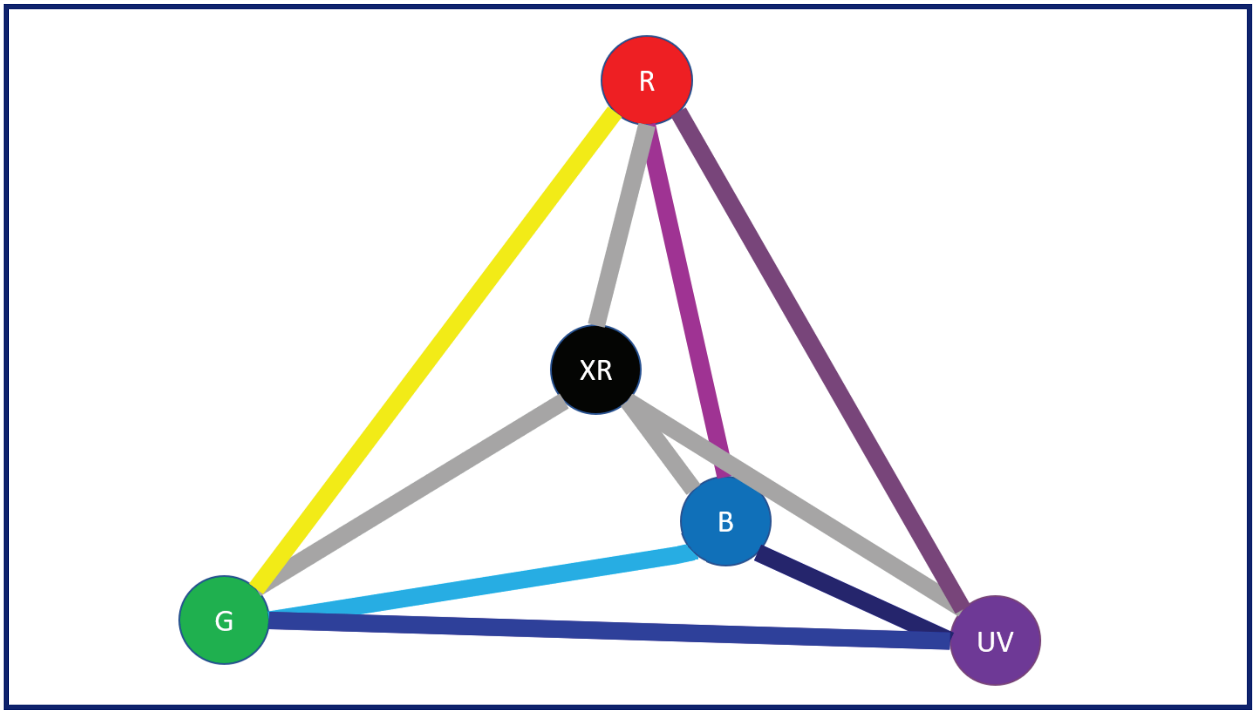


Figure 4. This three-dimensional projection of a four-dimensional hypertetrahedron shows how the basic color combinations for pentachromats (except white and black) can be represented. The vertices and edges represent the primary and secondary colors. The ten triangular faces correspond to the ten tertiary colors that pentachromats perceive, and there are five tetrahedrons that can be associated with each of the five quaternary colors of pentachromatic vision.

tetrahedrons. This is an artifact of flattening the hypertetrahedron into a three-dimensional form, just as three of the four triangular faces of the tetrahedron in Figure 3 appear as though they are inside of the triangle bound by R, G, and UV in the two-dimensional flattened version of the tetrahedron. Because we fully understand three spatial dimensions, we know that the fourth triangle bound by R, G, and UV in Figure 3 is a separate triangle with nothing in its internal region. But if we understood only two spatial dimensions, then we could simulate an internal region for that triangle by using the external region that is outside of the triangle, because that external region is still *separated* from the internal regions associated with the other triangles. In the same way, the three-dimensional external region outside of the tetrahedron bound by R, G, B, and UV in Figure 4 is a fifth region that can be colored with the quaternary color perception associated with simultaneous activation of Superman's R, G, B, and UV eye cones.

From the perspective of direct spatial understanding, the most difficult pentachromatic color-combination cases occur when k=0 and k=5. These cases have been saved for last because, as far as has been documented, not even Superman can directly imagine four spatial dimensions. According to the generalization, there should be "5 choose 0" = 1 K_0 subgraph, which is an empty graph, with no vertices or edges, that corresponds to the color black because zero eye cones are being activated by the zero colors chosen. In the lower dimensions, such as the triangle in Figure 2 or the tetrahedron in Figure 3, we were able to associate the color black with the external region outside of the triangle or tetrahedron. For the hypertetrahedron, we cannot see such an external region because it is only in four-dimensional space; but we can reason by analogy that it exists. Similarly, according to the generalization, there should be "5 choose 5" = 1 K_5 subgraph on the five primary-color vertices—which can be seen as true in Figure 4—and there should be a region that is inside of a boundary surface in four-dimensional space that contains *all five vertices*. We cannot directly see this boundary surface nor the internal region because we are looking at a flattened, three-dimensional representation of the hypertetrahedron in Figure 4, and this is analogous to not being able to perceive the surface of the tetrahedron in Figure 3 until we transform the two-dimensional, flattened diagram into a three-dimensional representation in our minds. Although Superman cannot, as far as we know, perform the same transformation of the diagram in Figure 4 into a four-dimensional mental representation, even we mere mortals can, once again, reason by analogy that there is a boundary surface in four-dimensional space that contains all five vertices and that separates the external region we associated with black from an internal region that we can associate with the color white because it represents simultaneous activation of all five of Superman's eye cones.

Topology Only Scratches the Surface

Because of the complexities introduced by higher dimensionality, it is natural to wonder whether the extra dimensionality is necessary. Is there, for example, a way to create a color sphere or similar surface for pentachromats analogous to the color sphere for tetrachromats in Figure 3? Keeping in mind that a sphere is the two-dimensional surface of a three-dimensional ball, the short answer is that complete graphs on five (or more) vertices can be embedded *only* onto a two-dimensional surface that is *more* complex than a sphere, but the embedding does not produce regions that can be associated with all combinations of three or more primary colors.

Due to Euler's formula relating the number of vertices, edges, and faces of planar graph embeddings,[16] there is no way to embed a K_5 onto a plane surface without crossing edges.[17] Since a plane can be mapped onto a sphere using a technique called stereographic projection,[18] there is also no way to embed K_5 onto a sphere without crossing edges. The absence of crossed edges is important because having crossed edges would mean that the face associated with two different cycles ("closed walks") of vertices and edges would overlap; it would not be possible to associate a single color with a face based on the vertices along its border.

In the branch of mathematics called topology, the complexity of (orientable) surfaces is measured by *genus*, which is the number of holes in the surface.[19] A sphere has genus 0, and the next simplest surface of genus 1 is a torus, which is the two-dimensional surface of a three-dimensional object that looks like a donut or a coffee mug with a handle. As shown in Figure 5, a K_5 can be embedded onto a torus without crossing any edges. However, the embedding has only four triangular faces, so the embedding does not create separate regions on the torus that could be colored with the other six of the ten tertiary colors in pentachromatic vision. Furthermore, although there are five other distinct ways to embed a K_5 onto a torus,[20] none have enough distinct faces on the torus for representing all tertiary colors. Similarly, the simplest non-orientable surface is the Möbius strip,[21] and there are two distinct ways to embed K_5 onto it,[22] neither of which contains enough faces to represent all tertiary colors of pentachromatic vision.

The limitations of surfaces described above for pentachromacy are representative of the general case for n-chromacy with $n \geq 5$. By selecting a sufficiently complex surface, K_n can be embedded with no crossing edges, so the primary colors can be displayed at vertex locations on the surface, and all secondary colors can be shown along the embeddings of the edges on the surface. However, a generalization of Euler's formula gives the limits based on surface complexity for how many faces an embedding in a surface can have,[23] and the limits are less than the binomial coefficient value for "n choose 3." Specifically, based on Euler's generalized formula, the number of faces F created by a graph embedding must be

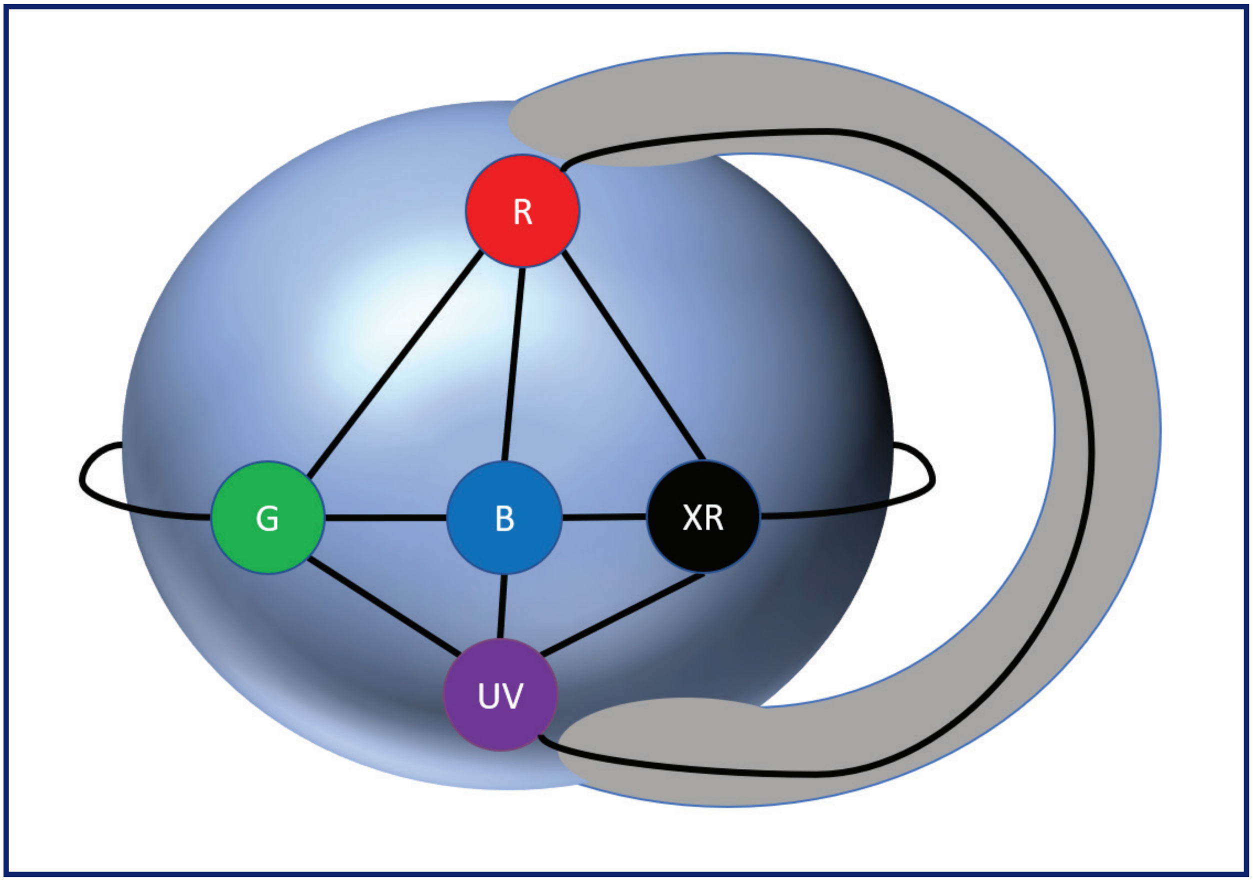


Figure 5. A complete graph on five vertices; i.e., a K_5, embedded on a torus (a sphere with a handle). This embedding shows that only four of the ten tertiary colors are represented by triangular faces of the embedding. This is one of only six distinct ways to embed a K_5 on a torus, each of which does not have enough faces to represent all tertiary colors.

less than or equal to $m - n + 2$, where m is the number of edges and n is the number of vertices in the graph. A complete graph K_n has n vertices and $n(n-1)/2$ edges. So, $F \leq (n(n-1)/2) - n + 2$. Meanwhile, for n-chromacy with $n \geq 5$, the number of tertiary colors is "n choose 3" = $n!/(3!(n-3)!) = (n(n-1)(n-2))/6$, which is greater than or equal to $n(n-1)/2$ when $n \geq 5$. Since, with $n \geq 5$, the number of tertiary colors in n-chromatic vision is at least $n(n-1)/2$, and there are, at most, $(n(n-1)/2) - n + 2$ faces in an embedding of K_n onto any surface, we can conclude that complete graph embeddings on two-dimensional surfaces do not have enough faces to associate with the tertiary colors of n-chromatic vision, not to mention the need for more faces to associate with the combinations of four or more colors.

Conclusion

There is a conspicuous line in *The Matrix* that expresses a commonplace gustatory concept in an unusual way: "… maybe they couldn't figure out what to make chicken taste like, which is why chicken tastes like everything."[24] The usual expression is the other way around, such as, "Don't be afraid to eat frog legs; they taste just like chicken." Relations such as "*A* tastes like *B*" are *symmetric* because the order of the operands *A* and *B* can be reversed.[25] If something tastes like chicken, then chicken tastes like that thing, too. The optical relation "*A* looks like *B*" is similarly symmetric.

In answer to the first question in the Introduction, the increased activation of our L-cones in the wavelength range of violet light—as well as the information loss intrinsic to our trichromatic color perception—are the combined reasons why red and blue look like purple and, symmetrically, why purple looks like red and blue. Makes one wonder what would have happened if Neo (in *The Matrix*) had taken both the red pill *and* the blue pill.[26]

The symmetric relation between purple and the combination of red and blue, which is implemented by our optical neural processing, maps the linear sequence of wavelengths of the visible light spectrum onto a color wheel. However, even if our L-cones were not more highly activated by violet light than by blue light, a color wheel would still have been a suitable structure for showing all primary and secondary color combinations of trichromatic vision because it has the same basic structure as a complete graph on three vertices drawn as a triangle.

In answer to the second question in the Introduction, a color sphere, or color tetrahedron, can be used to show the four primary colors, six secondary color combinations, and four tertiary color combinations of tetrachromatic vision. More generally, a complete graph on *n* vertices embedded as a hypertetrahedron in "*n*-1"-dimensional space is suitable for representing all basic color combinations of *n*-chromatic vision.

And now that you know this, there's no going back; so, delight in the knowledge and never wonder, "Why, oh, why didn't I take the *blue* pill?!"[27]

NOTES

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